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Tensor Products of Operator Spaces

DAVID P. BLECHER AND VERN I. PAULSEN*

Department of Mathematics, University of Houston, Houston, Texas 77204-3476

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In this paper we lay the foundations for a systematic study of tensor products of subspaces of C^*-algebras. To accomplish this, various notions of duality are introduced and employed. Elementary proofs of the complete injectivity of the Haagerup norm, and of the extension theorem for completely bounded maps, are given. Pisier's gamma norms are examined and found to be special cases of the Haagerup norm. We identify the greatest operator space cross norm and show that the spatial tensor norm is the least operator space cross norm in an appropriate sense. Indeed most of the elementary theory of Banach space tensor norms generalizes to the category of operator spaces. © 1991 Academic Press, Inc.

1. INTRODUCTION

In this paper we lay the foundations for a systematic study of tensor products of subspaces and subalgebras of C^*-algebras. It is now widely acknowledged that in order to fully understand subspaces of C^*-algebras it is necessary to take into account the matrix normed structure that they inherit from the containing C^*-algebra. However, to study tensor products of spaces one is naturally led to consider constructions involving their duals. For these reasons, our study of tensor products of subspaces of C^*-algebras leads us to consider several matrix norm structures on their duals. In general, these matrix normed dual spaces cannot be identified with subspaces of C^*-algebras completely isometrically, that is, in a manner which preserves all of their matrix norm structure. Thus, we are forced to study more general matrix normed spaces. However, there is a particular matrix normed structure on dual spaces which does allow these dual spaces to be identified completely isometrically with subspaces of C^*-algebras and this structure plays a central role in our considerations.

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In Section 2, we define the various matrix normed structures that we shall be concerned with and discuss some of the key features of each of these structures.

In Section 3, we study the Haagerup tensor norm of matrix normed spaces and give an elementary proof of the complete injectivity of this norm for tensor products of subspaces of C^*-algebras, that is, we show that tensor products of subspaces embed completely isometrically. Some of the main ideas of this proof can be extracted from Pisier's recent paper [Pi2], but they were discovered independently.

We give a new elementary proof of the Hahn–Banach extension theorem for completely bounded maps from subspaces of C^*-algebras into the $n \times n$ matrices, which makes essential use of the complete injectivity of the Haagerup tensor norm. Our proof of this result borrows ideas from Effros' proof [ER1, Bl1] but makes the role of the injectivity of the Haagerup norm more transparent.

To complete our examination of the Haagerup norm, we prove that it can be regarded as a "factorization" norm of the type studied in the Banach space tensor theory [Pi1]. This approach was suggested to the authors by Pisier.

In Section 4, we delineate further correspondences with the Banach space tensor theory. In particular, we show that Grothendieck's H-norm (also known as γ_2 [Pi1] and α_{22} [GL]) is just the Haagerup norm that one obtains by regarding Banach spaces as subspaces of C^*-algebras via the canonical embedding of a normed space into the C^*-algebra of continuous functions on the unit ball of its dual. More generally, we prove that Pisier's gamma tensor norms [Pi2], which include the H-norm, can each be regarded as the Haagerup tensor norm that one obtains via various "natural" representations of the Banach spaces involved as subspaces of C^*-algebras. Thus, Pisier's proof of the injectivity of these gamma norms follows from the injectivity of the Haagerup norm. Conversely, Pisier has shown that the Haagerup tensor norm can be regarded as a gamma tensor norm and hence the injectivity of the Haagerup norm follows from the injectivity of the gamma norms.

Thus as Banach space tensor norms these two theories are equivalent. However, as matrix tensor norms they are different and this subtle distinction has surprising consequences. For example, the Haagerup tensor norm is associative, while Grothendieck's H-norm is *not* associative. We explain this apparent paradox in Section 4.

Finally, in Section 5, we return to our main goal of describing matrix norm structures on tensor products of subspaces of C^*-algebras. Unlike tensor products of C^*-algebras, there are examples of cross norms on subspaces of C^*-algebras, which are smaller than the spatial tensor norm [ER2]. However, we prove that for any fixed pair of subspaces of

C^*-algebras, the spatial tensor norm is the minimal tensor norm which satisfies one additional condition. This additional condition requires that the induced norm on the tensor product of the dual spaces be a matricial cross norm when the duals are endowed with a particular dual matrix norm structure. This result is the analogue of the fact that the injective Banach space tensor norm, the λ-norm, is the minimal tensor norm among all tensor norms with the property that the induced norm on the tensor product of the dual spaces be a cross norm.

We identify the greatest matricial cross norm, which we call the projective operator space tensor norm. We show that the dual norm of the projective operator space norm is the spatial norm, and that a norm α lies between the spatial and the projective operator space norms if and only if α and the dual of α are matricial cross norms. Following the Banach space theory we examine the canonical correspondences. For instance, the spatial tensor product of two operator spaces X and Y is completely isometrically embedded in the operator space of completely bounded maps from X to Y' (or alternatively from Y to X'); and that the projective operator space tensor product of X and Y has a dual space which is completely isometrically isomorphic to the space of completely bounded maps from X to Y' (or alternatively from Y to X').

Finally, we define a notion of uniformity for operator space tensor norms and study its implications. This is the noncommutative analogue of Grothendieck's "reasonable" tensor norms [Gr, Ca2]. The natural examples of operator space tensor norms are uniform in this sense.

2. Matrix Normed Spaces

Let X be a vector space over $\mathbb{C}$ and let $M_{n,m}(X)$ denote the vector space of $n \times m$ matrices with entries from X, and let $M_{n,m} = M_{n,m}(\mathbb{C})$ be endowed with the norm that it inherits by regarding it as the linear transformations from the Hilbert space $\mathbb{C}^m$ to the Hilbert space $\mathbb{C}^n$. We call X a *matrix normed space* if each $M_{n,m}(X)$ is endowed with a norm $\|\cdot\|_{n,m}$ such that for A in $M_{n,p}$, B in $M_{p,q}(X)$, and C in $M_{q,m}$, we have $\|ABC\|_{n,m} \leqslant \|A\| \, \|B\|_{p,q} \, \|C\|$.

We set $M_n(X) = M_{n,n}(X)$. Assume we are given norms $\|\cdot\|_n$ on each $M_n(X)$ satisfying:

(i) For every B in $M_n(X)$, O in $M_m(X)$, $\|B \oplus O\|_{n+m} = \|B\|_n$,

(ii) For B in $M_n(X)$, A, C in M_n we have $\|ABC\|_n \leqslant \|A\| \, \|B\|_n \, \|C\|$.

If we assign norms to $M_{n,m}(X)$ by embedding it in $M_k(X)$, $k = \max\{n, m\}$, by adjoining rows or columns of 0's, then the resulting family of norms makes X a matrix normed space. For these reasons we shall often assign norms only to $M_n(X)$ and verify (i) and (ii) to construct a matrix normed space.

If X is a matrix normed space, then it is easily checked that each of the spaces $M_{r,s}(X)$ becomes matrix normed when we identify $M_{n,m}(M_{r,s}(X)) = M_{nr,ms}(X)$ in the canonical way.

If X is matrix normed, we set $M_{1,n}(X) = R_n(X)$, $M_{n,1}(X) = C_n(X)$, $R_n(\mathbb{C}) = R_n$, and $C_n(\mathbb{C}) = C_n$. Note that by the above remark these are matrix normed spaces.

If X and Y are matrix normed spaces and $\varphi: X \to Y$ is linear, then we define $\varphi^{(n)}: M_n(X) \to M_n(Y)$ via $\varphi^{(n)}((x_{ij})) = (\varphi(x_{ij}))$. We set $\|\varphi\|_{\mathrm{cb}} = \sup_n \|\varphi^{(n)}\|$ and say that φ is *completely bounded* when this number is finite. We say that φ is *completely contractive* if each $\varphi^{(n)}$ is contractive, and a *complete isometry* if each $\varphi^{(n)}$ is an isometry. Two matrix normed spaces are *completely isometrically isomorphic* if there is a complete isometry of the first space onto the second.

The spaces R_n, C_n, and $\mathbb{C}^n$ are isometrically isomorphic via the natural identification, but this map is not a complete isometry between R_n and C_n.

We now describe the principal matrix normed spaces that we shall be concerned with.

If $\mathscr{H}$ is a Hilbert space, set $\mathscr{H}^{(n)} = \mathscr{H} \oplus \cdots \oplus \mathscr{H}$ (n copies). The bounded linear operators on $\mathscr{H}$, $\mathscr{B}(\mathscr{H}) = \mathscr{B}(\mathscr{H}, \mathscr{H})$, becomes a matrix normed space if we identify $M_n(\mathscr{B}(\mathscr{H}))$ with $\mathscr{B}(\mathscr{H}^{(n)}, \mathscr{H}^{(n)})$ via $(L_{ij})(h_1, ..., h_n)^{\mathrm{t}} = (\sum_j L_{1j}(h_j), ..., \sum_j L_{nj}(h_j))^{\mathrm{t}}$ and endow each of these spaces with the operator norm. More generally, every C^*-algebra A is a matrix normed space when we identify $M_n(A)$ with $M_n \otimes A$ equipped with its unique C^*-norm. If A is a subalgebra of $\mathscr{B}(\mathscr{H})$, then this is just the norm that $M_n(A)$ inherits as a subspace of $M_n(\mathscr{B}(\mathscr{H}))$.

EXAMPLE 2.1. Every subspace of a C^*-algebra inherits a unique matrix normed structure from the C^*-algebra in which it is contained. Any matrix normed space which has a completely isometric (linear) representation as a subspace of a C^*-algebra we call an *operator space*. These matrix normed spaces have been characterized abstractly by Ruan [Ru] as the *L^∞-matrix normed spaces*. An L^∞-matrix normed space is a matrix normed space X such that if A is in $M_n(X)$ and B is in $M_m(X)$ then $\|A \oplus B\|_{n+m} = \max\{\|A\|_n, \|B\|_m\}$. Ruan proves that if X is an L^∞-matrix normed space then there is a completely isometric representation of X as a subspace of a C^*-algebra. Conversely, every operator space is easily seen to be an L^∞-matrix normed space.

EXAMPLE 2.2. If X is a matrix normed space, then X^{op} is the new matrix normed space that one obtains by setting $\|(x_{ij})\|_n^{\mathrm{op}} = \|(x_{ij})^{\mathrm{t}}\|_n$, where t denotes the transpose map. The identity map is an isometry between X and X^{op} but not necessarily a complete isometry. For example, R_n^{op} can be seen to be completely isometrically isomorphic to C_n.

EXAMPLE 2.3. If X is a normed space, then the canonical inclusion of X into the continuous functions on the unit ball of its dual allows us to identify X with a subspace of a C^*-algebra and hence endows X with a matrix normed structure such that X is an operator space [ER1]. We denote this operator space by $\mathrm{MIN}(X)$ and denote the norm on $M_n(\mathrm{MIN}(X))$ by $\|\cdot\|_n^{\min}$.

It is easily checked that the norm on $M_n(X) = M_n \otimes X$ obtained in this manner is the injective Banach space norm, $M_n \otimes_\lambda X$.

It is easily checked that if X is endowed with a matrix norm, and if $f\colon X \to \mathbb{C}$ is a contractive linear functional, then f is completely contractive. Hence

$$\|(x_{ij})\|_n^{\min} = \sup\{\|(f(x_{ij}))\| : \|f\| \leqslant 1\} \leqslant \|(x_{ij})\|_n.$$

Thus we see that $\mathrm{MIN}(X)$ is the minimum of all possible matrix norms on X, and this happens to be an operator space norm.

We shall call X a *commutative operator space* if X and $\mathrm{MIN}(X)$ are completely isometrically isomorphic, via the identity map. We remark here that if we are considering operator algebras this notation is less ambiguous than it may at first seem; in [Bl3] it is shown that a commutative operator space which is also a unital operator algebra is automatically a commutative algebra (in fact a uniform algebra).

EXAMPLE 2.4. Let X be a normed space and endow X with a matrix normed structure by setting $\|(x_{ij})\|_n^{\max} = \sup\{\|(\varphi(x_{ij}))\|\}$, where the supremum is over all Hilbert spaces $\mathscr{H}$, and over all contractive linear maps φ from X to $\mathscr{B}(\mathscr{H})$. The resulting matrix normed space is easily seen by Ruan's theorem [Ru] (or directly) to be an operator space and it is the largest of all operator space structures on X. We denote this space by $\mathrm{MAX}(X)$.

EXAMPLE 2.5. Let X be a normed space and define a norm on $M_n(X) = M_n \otimes X$ by endowing this space with the projective tensor norm. It is easily seen that X becomes a matrix normed space, which we denote $\mathrm{PROJ}(X)$, and that this matrix norm is the largest of all matrix norms on X which happen to be cross-norms on $M_n \otimes X$. However, in general, a matrix norm on X need not be a cross norm (see Example 2.6). For $A = (A_{ij})$ in M_n and x in X, we only have that

$$\|A \otimes x\|_n = \|(A_{ij}x)\|_n \geqslant \|A\| \, \|x\|,$$

for an arbitrary matrix norm.

In general, the projective matrix norms are larger than the MAX matrix norms and smaller than the maximum of all matrix norms. PROJ(X) is not in general an operator space.

EXAMPLE 2.6. Let X be a matrix normed space, and let X' be the dual space. We can identify $M_n(X')$ with $M_n(X)'$ via the duality, $\langle (f_{ij}), (x_{ij}) \rangle = \sum_{ij} f_{ij}(x_{ij})$, for f_{ij} in X' and x_{ij} in X. This induces a norm on $M_n(X')$ for all n, and it is easily checked that this is a matrix norm on X' (see [CE]). We shall call this matrix normed space the *Choi–Effros dual* of X and denote it by X'_{ce}.

Note that for f in X', the $n \times n$ matrix with diagonal entry f, Diag(f), satisfies

$$\|\mathrm{Diag}(f)\|_n^{\mathrm{ce}} = n\ \|f\|.$$

Thus, this matrix norm on $M_n(X')$ is not a cross-norm.

EXAMPLE 2.7. Let X be a matrix normed space. We define the *left dual of* X, X'_{l}, to be the dual of X together with the norms on $M_n(X')$ obtained by identifying $M_n(X')$ with $\mathscr{B}(C_n(X), C_n)$, the bounded maps from $C_n(X)$ to C_n, via

$$(f_{ij})(x_1, ..., x_n)^{\mathrm{t}} = \left(\sum_j f_{1j}(x_j), ..., \sum_j f_{nj}(x_j)\right)^{\mathrm{t}}.$$

It is easily checked that these are matrix norms.

Similarly, the *right dual of* X, X'_{r}, is the matrix normed space obtained by identifying $M_n(X')$ with $\mathscr{B}(R_n(X), R_n)$ via matrix multiplication on the right,

$$(x_1, ..., x_n)(f_{ij}) = \left(\sum_i f_{i1}(x_i), ..., \sum_i f_{in}(x_i)\right).$$

With these identifications, it is easily checked that the dot product pairing defines a completely isometric isomorphism between $(C_n)'_{\mathrm{l}}$ and R_n, and between $(R_n)'_{\mathrm{r}}$ and C_n.

EXAMPLE 2.8. Let X and Y be matrix normed spaces, and let $B(X, Y)$ denote the bounded linear maps from X to Y. We define the *left matrix norm structure* on $B(X, Y)$ by identifying (L_{ij}) in $M_n(B(X, Y))$ with the map in $B(M_n(X), M_n(Y))$ defined by matrix multiplication on the left, that is,

$$(L_{ij})(x_{ij}) = \left(\sum_k L_{ik}(x_{kj})\right).$$

We have endowed $B(X, Y)$ with a matrix norm structure which we denote by $B_l(X, Y)$. Now although $B_l(X, Y)$ is not a matrix norm space in general (in the strict sense of the definition at the beginning of this section) this will not matter. In fact in all of our applications it will actually be an operator space. In a similar fashion, we define the *right matrix norm* structure $B_r(X, Y)$ by identifying (L_{ij}) with a map in $B(M_n(X), M_n(Y))$ via multiplication on the right,

$$(x_{ij})(L_{ij}) = \left(\sum_k L_{kj}(x_{ik})\right).$$

Note that the identity map is a complete isometry from $B_r(X, Y)$ to $B_l(X^{\text{op}}, Y^{\text{op}})^{\text{op}}$.

We remark that B_l (and B_r) are simple versions of another interesting construction Π_l (and Π_r). We define $\Pi_l(X, Y)$ to be the set of linear operators $T: X \to Y$ such that there is a constant C with $\|(Tx_1, Tx_2, ..., Tx_n)\|_{1,n} \leqslant C\,\|(x_1, ..., x_n)\|_{1,n}$, for all n and $x_1, ..., x_n \in X$. We define $M_n(\Pi_l(X, Y))$ similarly to $M_n(B_l(X, Y))$, except we allow the matrix $[L_{ij}]$ in $M_n(\Pi_l(X, Y))$ to act on the left on $n \times k$ matrices with entries in X, k arbitrary. We omit the calculation that this defines a norm on $M_n(\Pi_l(X, Y))$. With this matrix norm structure $\Pi_l(X, Y)$ is a matrix normed space in the strict sense. In our applications we have $\Pi_l(X, Y) = B_l(X, Y)$ completely isometrically, so we use the simpler version. The above statements follow analogously for $\Pi_r(X, Y)$.

PROPOSITION 2.9. *Let X be a matrix normed space; then the identification of X' with $B(X, \mathbb{C})$ defines a completely isometric isomorphism between X'_l and $B_l(X, \mathbb{C})$ and between X'_r and $B_r(X, \mathbb{C})$.*

Proof. We consider only the left case. We need to verify that for any matrix of linear functionals (f_{ij}),

$$\text{(1)} \qquad \sup\left\{\sum_i \left|\sum_j f_{ij}(x_j)\right|^2 : \|(x_1, ..., x_n)^t\|_{n,1} \leqslant 1\right\},$$

and

$$\text{(2)} \qquad \sup\left\{\left\|\left(\sum_k f_{ik}(x_{kj})\right)\right\|^2 : \|(x_{ij})\|_n \leqslant 1\right\},$$

are equal.

Clearly, (1) is smaller than (2), since it is (2) restricted to matrices with only the first column non-zero. On the other hand,

$$(3)\quad \left\| \left(\sum_k f_{ik}(x_{kj}) \right) \right\|^2$$
$$= \sup \left\{ \sum_i \left| \sum_{k,j} f_{ik}(x_{kj})\, \lambda_j \right|^2 : |\lambda_1|^2 + \cdots + |\lambda_n|^2 \leqslant 1 \right\}.$$

Set $x'_k = \sum_j x_{kj} \lambda_j$ and note that $\|(x'_1, ..., x'_n)^t\| \leqslant 1$. Thus, the right-hand side of (3) is a supremum of terms appearing in (1), from which the other inequality follows. ∎

EXAMPLE 2.10. Let X and Y be matrix normed spaces and let $\mathrm{CB}(X, Y)$ denote the completely bounded maps from X to Y. For (L_{ij}) in $M_n(\mathrm{CB}(X, Y))$ and (x_{kl}) in $M_m(X)$ consider $(L_{ij}(x_{kl}))$ in $M_m(M_n(Y)) = M_{mn}(Y)$. We set

$$\|(L_{ij})\|_n = \sup\{ \|(L_{ij}(x_{kl}))\|_{mn} : \|(x_{kl})\|_m \leqslant 1 \}.$$

It is not difficult to check that with these definitions $\mathrm{CB}(X, Y)$ becomes a matrix normed space. Note also that if we identify $M_n(\mathrm{CB}(X, Y))$ with $\mathrm{CB}(X, M_n(Y))$ by identifying (L_{ij}) with the map $L: X \to M_n(Y)$ given by $L(x) = (L_{ij}(x))$, then $\|(L_{ij})\|_n = \|L\|_{\mathrm{cb}}$.

It is important to note if Y is an operator space, then $\mathrm{CB}(X, Y)$ is easily seen to be an L^∞-matrix normed space, and so by Ruan's theorem $\mathrm{CB}(X, Y)$ is completely isometrically isomorphic to an operator space. This may also be seen directly, as in [B14]. After this paper had been written the authors discovered that this definition had been given in [ER3].

When $Y = \mathbb{C}$, since every bounded linear functional is completely bounded with the same norm, we have that X' and $\mathrm{CB}(X, \mathbb{C})$ are isometrically isomorphic. This identification endows X' with a matrix normed structure such that the dual of X becomes an operator space. We call this the *standard operator space dual* of X, and denote it simply by X'.

The intrinsic value of this dual is that it allows us to stay in the category of operator spaces. Moreover, many constructions on the category of operator spaces transform naturally with this duality. For example, if X and Y are operator spaces and $T: X \to Y$ is completely bounded, then the adjoint map $T': Y' \to X'$ is completely bounded and $\|T'\|_{\mathrm{cb}} = \|T\|_{\mathrm{cb}}$. It is a consequence of the Hahn–Banach extension theorem for completely bounded maps that if X and Y are operator spaces and $i: X \to Y$ is a completely isometric inclusion, then $i'': X'' \to Y''$ is a completely isometric inclusion. A more difficult result involving this operator space dual is a characterization of the space $\mathrm{MAX}(X)$ for a normed space X. It turns out that $\mathrm{MIN}(X)' = \mathrm{MAX}(X')$ and $\mathrm{MAX}(X)' = \mathrm{MIN}(X')$, but these results are beyond the scope of this paper. (See [Bl4]). One important point to note is that if X is a matrix normed space, which is *not* an operator space, then the canonical embedding of X into X'' cannot be a complete isometry since X'' is an operator space. However, this is the only obstruction.

THEOREM 2.11. *Let X be an operator space. Then the canonical embedding of X into $X'' = \mathrm{CB}(\mathrm{CB}(X, \mathbb{C}), \mathbb{C})$ is a complete isometry.*

Proof. We may assume that X is contained in $B(\mathscr{H})$ for some Hilbert space $\mathscr{H}$.

Let $\mathscr{F}$ be the family of finite dimensional subspaces F of $\mathscr{H}$; for $F \in \mathscr{F}$ write φ_F for the compression map from $B(\mathscr{H})$ to $B(F)$.

Now if $\hat{}: X \to X''$ is the canonical embedding we see that if $[x_{ij}] \in M_n(X)$ then

$$\begin{aligned}\|[\hat{x}_{ij}]\|_n &= \sup\{\|[\hat{x}_{ij}(f_{kl})]\| : [f_{kl}] \in \mathrm{BALL}(M_m(X'))\}\\ &= \sup\{\|[f_{kl}(x_{ij})]\| : [f_{kl}] \in \mathrm{BALL}(M_m(X'))\}\\ &= \sup\{\|T_n([x_{ij}])\| : T \in \mathrm{BALL}(\mathrm{CB}(X, M_m))\}.\end{aligned}$$

Certainly this last quantity is less than or equal to $\|[x_{ij}]\|_n$. Now $\|[x_{ij}]\| = \sup\{\|(\varphi_F)_n([x_{ij}])\| : F \in \mathscr{F}\}$ (this is obvious: consider sets F of vectors on which $[x_{ij}]$ nearly achieves its norm). Since each φ_F may be regarded as a completely contractive map into some M_m (where $m = \dim(F)$) the result follows. ∎

In [Bl4] it is shown that $X'' = (X'_{\mathrm{ce}})'_{\mathrm{ce}}$ completely isometrically.

3. THE HAAGERUP NORM

In this section we develop the properties of the Haagerup tensor norm. We give an elementary proof that for operator spaces this tensor norm is completely injective. Using some duality considerations and the complete injectivity of the Haagerup norm we give another proof of the extension theorem for completely bounded maps into M_n. Finally, we show (for operator spaces) how to realize the Haagerup norm as a factorization norm.

Let $A = (x_{ij})$ be in $M_{n,k}(X)$ and $B = (y_{ij})$ be in $M_{k,m}(Y)$; then $A \odot B$ is the element of $M_{n,m}(X \otimes Y)$ given by

$$A \odot B = \left(\sum_{r=1}^{k} x_{ir} \otimes y_{rj}\right).$$

This operation was first introduced by Effros [Ef]. It is valuable to note that if L is a matrix of scalars, then $A \odot (LB) = (AL) \odot B$.

If X and Y are matrix normed spaces then we define the *Haagerup matrix norm* on $X \otimes Y$ by setting for U in $M_n(X \otimes Y)$,

$$\|U\|_h = \inf\left\{\sum_{k=1}^{m} \|A_k\| \, \|B_k\| : U = \sum_{k=1}^{m} A_k \odot B_k\right\},$$

where the infimum is taken over all such expressions for U with A_k in $M_{n,n_k}(X)$, B_k in $M_{n_k,n}(Y)$. It is not difficult to see that with these definitions $X \otimes Y$ becomes a matrix normed space, which we denote $X \otimes_h Y$.

The following is easily verified, and we leave its proof to the reader.

PROPOSITION 3.1. *The Haagerup matrix tensor norm is associative. That is, if X, Y, and Z are matrix normed spaces, then $(X \otimes_{\mathrm{h}} Y) \otimes_{\mathrm{h}} Z$ and $X \otimes_{\mathrm{h}} (Y \otimes_{\mathrm{h}} Z)$ are completely isometrically isomorphic. Moreover, for U in $M_n(X \otimes Y \otimes Z)$ this norm is given by*

$$\|U\|_{\mathrm{h}} = \inf\left\{\sum_{k=1}^{m} \|A_k\| \, \|B_k\| \, \|C_k\| : U = \sum_{k=1}^{m} A_k \odot B_k \odot C_k\right\}.$$

There are many instances in which the sums appearing in the definition of the Haagerup norm can be avoided.

We call a matrix normed space X *2-row summing* if for $x = (x_1, ..., x_k)$ in $R_k(X)$, $y = (x_{k+1}, ..., x_{k+j})$ in $R_j(X)$, then $z = (x_1, ..., x_{k+j})$ in $R_{k+j}(X)$ satisfies $\|z\|^2 \leqslant \|x\|^2 + \|y\|^2$. We define *2-column summing* analoguously. We say that X is *2-summing* if it is both 2-row and 2-column summing. Operator spaces are easily seen to be 2-summing. If X and Y are matrix normed then $B_1(X, Y)$ is a 2-column summing space, which is generally not an operator space. The importance of these classes of spaces can be seen below and in Theorem 3.4.

LEMMA 3.2. *If X is 2-row summing, and if Y is 2-column summing, then for u in $X \otimes Y$,*

$$\|u\|_{\mathrm{h}} = \inf\{\|A\| \, \|B\| : u = A \odot B\}.$$

Proof. Note that if $u = \sum_{i=1}^{k} A_i \odot B_i$ with A_i in $M_{1,n_i}(X)$, B_i in $M_{n_i,1}(Y)$, then $u = A \odot B$, where $A = (r_1 A_1, ..., r_k A_k)$ is in $M_{1,n}(X)$, $B = (r_1^{-1} B_1, ..., r_k^{-1} B_k)$ is in $M_{n,1}(Y)$, $n = n_1 + \cdots + n_k$, and $r_i > 0$. Moreover, by the hypotheses

$$\|A\|^2 \, \|B\|^2 \leqslant \left(\sum_{i=1}^{k} r_i^2 \, \|A_i\|^2\right)\left(\sum_{i=1}^{k} r_i^{-2} \, \|B_i\|^2\right).$$

Choosing $r_i = \|B_i\|/\|A_i\|$, we have that $\|A\| \, \|B\| \leqslant \sum_{i=1}^{k} \|A_i\| \, \|B_i\|$, from which it follows that the infimum in the definition of $\|u\|_h$ is achieved over sums of length 1. ∎

We now turn our attention to the proof of the complete injectivity of the Haagerup norm. First, we need to establish some elementary facts about tensors. For $u \in X \otimes Y$, and $u = \sum_{i=1}^{n} x_i \otimes y_i$, let $E = \operatorname{span}\{x_1, ..., x_n\}$, $F = \operatorname{span}\{y_1, ..., y_n\}$. If we choose a subset of $\{y_1, ..., y_n\}$ which is a basis for F, $y_1, ..., y_k$ (after reordering), then after expressing $y_{k+1}, ..., y_n$ as linear combinations of $y_1, ..., y_k$ we may write $u = \sum_{i=1}^{k} x_i' \otimes y_i$ with x_i' in E. Choosing a basis $x_1', ..., x_m'$ for the span of $\{x_1', ..., x_k'\}$ we may rewrite $u = \sum_{i=1}^{m} x_i' \otimes y_i'$. Clearly, after finitely many iterations of this process we will have expressed $u = \sum_{i=1}^{p} \hat{x}_i \otimes \hat{y}_i$ with $\{\hat{x}_1, ..., \hat{x}_p\}$ and $\{\hat{y}_1, ..., \hat{y}_m\}$ linearly independent subsets of E and F, respectively. (In fact, the sets $\{x_1', ..., x_m'\}$, $\{y_1', ..., y_m'\}$ are already linearly independent.)

The following is undoubtedly well known.

LEMMA 3.3. *Let $u \in X \otimes Y$ and let $u = \sum_{i=1}^{n} x_i \otimes y_i = \sum_{j=1}^{k} v_j \otimes w_j$ be two ways to express u as a sum of elementary tensors such that each of the sets $\{x_1, ..., x_n\}$, $\{y_1, ..., y_n\}$, $\{v_1, ..., v_m\}$, and $\{w_1, ..., w_m\}$ is linearly independent. Then* $\operatorname{span}\{x_1, ..., x_n\} = \operatorname{span}\{v_1, ..., v_m\}$, *and* $\operatorname{span}\{y_1, ..., v_n\} = \operatorname{span}\{w_1, ..., w_m\}$.

Proof. Let $f_i \colon X \to \mathbb{C}$ be linear functionals with $f_i(x_j) = \delta_{ij}$, $i, j = 1, ..., n$. For $f_i \otimes 1 \colon X \otimes Y \to Y$, we have that $y_i = (f_i \otimes 1)(u) = \sum_{k=1}^{m} f_i(v_k)\, w_k$, from which it follows that $y_i \in \operatorname{span}\{w_1, ..., w_m\}$. The remaining cases follow similarly. ∎

Note that by the above $m = n$ and this value we call the *rank* of u.

THEOREM 3.4. *Let X and Y be matrix normed spaces with X row 2-summing and Y column 2-summing. If $X_1 \subseteq X$ and $Y_1 \subseteq Y$ are subspaces, then the canonical inclusion of $X_1 \otimes_{\mathrm{h}} Y_1$ into $X \otimes_{\mathrm{h}} Y$ is an isometry.*

Proof. Let $\|u\|_1$ denote the norm of u in $X_1 \otimes_{\mathrm{h}} Y_1$ and let $\|u\| = \inf\{\|A\| \, \|B\| : u = A \odot B\}$, $A \in M_{1,n}(X)$, $B \in M_{n,1}(Y)$ denote the norm of u in $X \otimes_{\mathrm{h}} Y$. Clearly, $\|u\| \leqslant \|u\|_1$.

Write $u = A \odot B$, $A = (x_1, ..., x_n) \in M_{1,n}(X)$, $B = (y_1, ..., y_n)^{\mathrm{t}} \in M_{n,1}(Y)$ and assume that $y_1, ..., y_k$ are linearly independent. Thus, we may write $B = L(y_1, ..., y_k)^{\mathrm{t}}$, where L is an $n \times k$ matrix of scalars. If we let $L = UP$ be the polar form of L, then U is an $n \times k$ isometry and P is a $k \times k$ invertible matrix. Let $B' = (y_1', ..., y_k')^{\mathrm{t}} = P(y_1, ..., y_k)^{\mathrm{t}}$, so that $u = A \odot (UB') = (AU) \odot B'$. Note that

$$\|A\| \, \|B\| = \|A\| \, \|UB'\| = \|A\| \, \|B'\| \geqslant \|AU\| \, \|B'\|.$$

Thus, we have written $u = A' \odot B'$ with $\|A'\|\ \|B'\| \leqslant \|A\|\ \|B\|$ and with the entries of B' linearly independent. Inductively, we may write $u = A'' \odot B''$ such that $\|A''\|\ \|B''\| \leqslant \|A\|\ \|B\|$ with the entries of A'' and of B'' linearly independent. (Again two iterations suffice.)

Since $u \in X \otimes Y$, by Lemma 3.3 we see that the entries of A'' and of B'' belong to X_1 and Y_1, respectively. Hence $\|u\|_1 \leqslant \|A''\|\ \|B''\| \leqslant \|A\|\ \|B\|$ and since A and B were arbitrary, $\|u\|_1 \leqslant \|u\|$. ∎

It is interesting to note that the only property of the matrix normed spaces used in the proof of the injectivity of the Haagerup norm was that $\|u\|_{\mathrm{h}} = \inf\{\|A\|\ \|B\| : u = A \odot B\}$, with no sums needed. In spite of this fact we shall see later that there are matrix normed spaces for which the Haagerup norm is *not* injective.

PROPOSITION 3.5. *Let X and Y be matrix normed spaces. Then the map Φ: $C_n(X) \otimes_{\mathrm{h}} R_n(Y) \to M_n(X \otimes_{\mathrm{h}} Y)$ given by $\Phi(A \otimes B) = A \odot B$ is a completely isometric isomorphism.*

Proof. Let $U = \sum_{i=1}^{k} A_i \otimes B_i = (A_1, ..., A_k) \odot (B_1, ..., B_k)^{\mathrm{t}}$; then $\Phi(U) = \sum_{i=1}^{k} A_i \odot B_i$. If we regard $V = (A_1, ..., A_k)$ as an element of $M_{n,k}(X)$ with columns A_i, and $W = (B_1, ..., B_k)^{\mathrm{t}}$ as an element of $M_{k,n}(Y)$ with rows B_i then $\Phi(U) = V \odot W$.

Since $\|(A_1, ..., A_k)\|$ in $M_{1,k}(C_n(X))$ is the same as $\|V\|$ in $M_{n,k}(X)$, and $\|(B_1, ..., B_k)^{\mathrm{t}}\|$ in $M_{k,1}(R_n(Y))$ is the same as $\|W\|$ in $M_{k,n}(Y)$, it follows that $\|U\| = \|\Phi(U)\|_h$, and so Φ is an isometry. The proof that Φ is a complete isometry is identical. ∎

THEOREM 3.6. *Let X and Y be operator spaces, $X_1 \subseteq X$, $Y_1 \subseteq Y$ subspaces. Then the inclusion of $X_1 \otimes_{\mathrm{h}} Y_1$ into $X \otimes_{\mathrm{h}} Y$ is a complete isometry.*

Proof. The inclusion is an isometry since operator spaces are 2-summing, and a complete isometry by Proposition 3.5 and the fact that $C_n(X)$ and $R_n(Y)$ are operator spaces and hence 2-summing. ∎

We now focus on some extension theorems for completely bounded maps which follow from the complete injectivity of the Haagerup tensor norm and some elementary duality considerations.

Let X, Y, and Z be matrix normed spaces, let $\varphi: X \otimes_{\mathrm{h}} Y \to Z$ be completely bounded and define $L_\varphi: X \to B_{\mathrm{l}}(Y, Z)$ and $R_\varphi: Y \to B_{\mathrm{r}}(X, Z)$ by $L_\varphi(x)(y) = \varphi(x \otimes y)$ and $R_\varphi(y)(x) = \varphi(x \otimes y)$. Note that for A in $M_n(X)$ and B in $M_n(Y)$ we have that

$$\varphi^{(n)}(A \odot B) = (L_\varphi^{(n)}(A))(B) = (R_\varphi^{(n)}(B))(A),$$

from which it follows that $\|\varphi\|_{\mathrm{cb}} = \|L_\varphi\|_{\mathrm{cb}} = \|R_\varphi\|_{\mathrm{cb}}$. We have the following:

PROPOSITION 3.7. *Let X, Y, and Z be matrix normed spaces. Then the maps $\varphi \to L_\varphi$ and $\varphi \to R_\varphi$ define isometries from the space* $\mathrm{CB}(X \otimes_{\mathrm{h}} Y, Z)$ *onto* $\mathrm{CB}(X, B_{\mathrm{l}}(Y, Z))$ *and onto* $\mathrm{CB}(Y, B_{\mathrm{r}}(X, Z))$ *respectively.*

We remark that $\mathrm{CB}(X \otimes_{\mathrm{h}} Y, Z) = \mathrm{CB}(X, \Pi_{\mathrm{r}}(Y, Z)) = \mathrm{CB}(Y, \Pi_{\mathrm{l}}(X, Z))$ isometrically too. We omit the proof.

With these identifications, a proof that every completely bounded map from an operator space into M_n extends (see [Pa1, Wi]) becomes an easy exercise in bootstrapping.

THEOREM 3.8. *Let X be an operator space, and let $Y \subseteq X$ be a subspace. Then every completely bounded map $\varphi\colon Y \to M_n$ can be extended to a completely bounded map $\tilde{\varphi}\colon X \to M_n$ with*

$$\|\varphi\|_{\mathrm{cb}} = \|\tilde{\varphi}\|_{\mathrm{cb}}.$$

Proof. Since R_n and X are operator spaces $R_n \otimes_{\mathrm{h}} Y$ is completely isometrically included in $R_n \otimes_{\mathrm{h}} X$ by Theorem 3.6. Hence every bounded linear functional on $R_n \otimes_{\mathrm{h}} Y$ extends to a bounded linear functional on $R_n \otimes_{\mathrm{h}} X$. Using the fact that bounded linear functionals are completely bounded, and the identifications (see Proposition 3.7) of $\mathrm{CB}(R_n \otimes_{\mathrm{h}} Y, \mathbb{C})$ with $\mathrm{CB}(Y, B_{\mathrm{r}}(R_n, \mathbb{C}))$ and $\mathrm{CB}(R_n \otimes_{\mathrm{h}} X, \mathbb{C})$ with $\mathrm{CB}(X, B_{\mathrm{r}}(R_n, \mathbb{C}))$, we see that every completely bounded linear map from Y to $B_{\mathrm{r}}(R_n, \mathbb{C})$ extends to a map from X to $B_{\mathrm{r}}(R_n, \mathbb{C})$ with the same cb-norm.

It is easy to see that the dot product pairing between R_n and C_n defines a completely isometric isomorphism of $B_{\mathrm{r}}(R_n, \mathbb{C})$ onto C_n. Thus we have proven an extension theorem for completely bounded maps into C_n.

In [PS] it is shown that the Haagerup tensor product of operator spaces is again an operator space (see also [Ru]). Now consider the operator spaces $Y \otimes_{\mathrm{h}} C_n \subseteq X \otimes_{\mathrm{h}} C_n$. By the above argument every map in $\mathrm{CB}(Y \otimes_{\mathrm{h}} C_n, C_n)$ extends to a map in $\mathrm{CB}(X \otimes_{\mathrm{h}} C_n, C_n)$. However, using Proposition 3.7 again we see that $\mathrm{CB}(Y \otimes_{\mathrm{h}} C_n, C_n) = \mathrm{CB}(Y, B_{\mathrm{l}}(C_n, C_n))$ isometrically, and the same is true with Y replaced by X. Repeating the argument above we conclude that a completely bounded map from Y into $B_{\mathrm{l}}(C_n, C_n)$ extends to a completely bounded map from X into $B_{\mathrm{l}}(C_n, C_n)$ with the same cb-norm.

Finally, the proof is completed by observing that $B_{\mathrm{l}}(C_n, C_n) = M_n$ completely isometrically, via the usual identification of a matrix with a linear transformation. ∎

Remark 3.9. The only non-elementary point in the above proof is the fact that if X is an operator space, then $X \otimes_{\mathrm{h}} C_n$ is again an operator space. The only known proofs of this fact rest ultimately on the characterization of operator systems given in [CE]. However, we can avoid using this result by working with L^∞-matrix normed spaces throughout. It is not hard to see that if X is L^∞-matrix normed then X, $C_n(X)$, and $R_n(X)$ are 2-summing. Thus the proof of Theorem 3.6 shows, independently of Ruan's theorem, that the Haagerup tensor product behaves completely injectively for L^∞-matrix normed spaces. It is also elementary that if X and Y are L^∞-matrix normed spaces, then so is $X \otimes_{\mathrm{h}} Y$. Thus using these observations one may indeed recover Theorem 3.8 (and 3.6) without using results in [CE], [PS], or [Ru], since of course every operator space is an L^∞-matrix normed space.

Remark 3.10. If the Haagerup tensor norm was completely injective for all matrix normed spaces, then the techniques of the proof of Theorem 3.8 would apply to yield that completely bounded maps into M_n enjoy the extension property for all matrix normed spaces. However, in [ER1] examples were given of a matrix normed space, a subspace, and a completely bounded map of the subspace into M_n which did not possess completely bounded extensions of the same cb-norm. Thus, there must be examples of matrix normed spaces such that the Haagerup tensor is not completely injective. It is interesting to recall that all that is really essential to the proof of the complete injectivity of the Haagerup norm for operator spaces is that the norm be achieved using only sums of length one.

Proposition 3.7, combined with the complete injectivity of the Haagerup tensor norm, can be used to generate many exotic matrix normed spaces X with the Hahn–Banach extension property for completely bounded maps from operator spaces into X. Proposition 3.7 is also useful for identifying dual pairings.

Note that by Proposition 3.4, $M_n = C_n \otimes_{\mathrm{h}} R_n$. On the other hand, by Proposition 3.7, $\mathrm{CB}(R_n \otimes_{\mathrm{h}} C_n, \mathbb{C}) = \mathrm{CB}(R_n, B_1(C_n, \mathbb{C})) = \mathrm{CB}(R_n, R_n) = M_n$, where the equals sign denotes isometries. Thus $R_n \otimes_{\mathrm{h}} C_n$ can be isometrically identified with the dual of M_n, $(M_n', \|\cdot\|_1)$ where $\|\cdot\|_1$ denotes the trace class norm.

If we let $\Psi\colon R_n \otimes_{\mathrm{h}} C_n \to M_n'$ denote this map, then for $u = \sum_i r_i \otimes c_i$, $\Psi(u) = \Phi_u$, where $\Phi_u(A) = \sum_i r_i A c_i$. Identify $M_k(R_n \otimes_{\mathrm{h}} C_n) = C_k(R_n) \otimes_{\mathrm{h}} R_k(C_n) = M_{k,n} \otimes_{\mathrm{h}} M_{n,k}$ and $M_k(M_n') = \mathrm{CB}(M_n, M_k)$; then the map Ψ_k becomes $\Psi_k(U) = \Phi_U\colon M_n \to M_k$, where $\Phi_U(A) = \sum_i X_i A Y_i$ if $U = \sum X_i \otimes Y_i$ $(X_i \in M_{k,n}, Y_i \in M_{n,k})$. Using the generalized Stinespring representation of completely bounded maps, we see that $\|\Phi_U\|_{\mathrm{cb}} = \|U\|_{\mathrm{h}}$ and so Ψ is a complete isometry.

We now turn our attention to factorization norms. If $u = \sum x_i \otimes y_i$ in $X \otimes Y$, then we may identify u with a map $\tilde{u}\colon X' \to Y$ via $\tilde{u}(f) = \sum f(x_i)\, y_i$. If we fix a Banach space Z, then by considering factorization through Z, we may induce a norm on $X \otimes Y$, namely $\|u\| = \inf\{\sum_j \|S_j\|\, \|T_j\|\}$, where the infimum is over all representations, $\tilde{u} = \sum_j T_j S_j$, where $S_j\colon X' \to Z$ and

$T_j: Z \to Y$. Factorization norms play a central role in the theory of tensor products of Banach spaces (see [Pi1, GL]).

THEOREM 3.11. *Let X and Y be operator spaces and let $u \in X \otimes_{\mathrm{h}} Y$; then*

$$\|u\|_{\mathrm{h}} = \inf\{\|S\|_{\mathrm{cb}}\, \|T\|_{\mathrm{cb}} : S \in \mathrm{CB}(X', R_n),\, T \in \mathrm{CB}(R_n, Y),\, \tilde{u} = TS\}.$$

Proof. Let $u = A \odot B$, $A = (x_1, ..., x_n)$, $B = (y_1, ..., y_n)^{\mathrm{t}}$ and define $S: X' \to R_n$ and $T: R_n \to Y$ by $S(f) = [f(x_1), ..., f(x_n)]$, $T([\lambda_1, ..., \lambda_n]) = \lambda_1 y_1 + \cdots + \lambda_n y_n$, so that $\tilde{u} = TS$, $S \in \mathrm{CB}(X', R_n)$, $T \in \mathrm{CB}(R_n, Y)$. We shall show that $\|S\|_{\mathrm{cb}} = \|A\|$, $\|T\|_{\mathrm{cb}} = \|B\|$, from which it follows that $\|u\|_{\mathrm{h}}$ is larger than the right-hand side of the above equation.

Note that $T((\lambda_1, ..., \lambda_n)) = (\lambda_1, ..., \lambda_n) B$, from which it follows that $\|T\|_{\mathrm{cb}} \leqslant \|B\|$. On the other hand, if we define $C = (C_{ij})$ in $M_n(R_n)$ via $C_i = e_i$, $C_{ij} = 0$, $j \neq 1$, then $\|C\| = 1$, and $\|T^{(n)}(C)\| = \|B\|$. Thus $\|T\|_{\mathrm{cb}} = \|B\|$.

For the other equality, define $A_1 = (a_{kl})$ in $M_n(X)$ by $a_{1l} = x_l$, $a_{kl} = 0$ for $k \neq 1$, so that $\|A\| = \|A_1\|$.

By Theorem 2.11,

$$\begin{aligned}\|A_1\| &= \sup\{\|(f_{ij}(a_{kl}))\| : \|(f_{ij})\| \leqslant 1\}\\ &= \sup\{\|(S(f_{ij}))\| : \|(f_{ij})\| \leqslant 1\}\\ &= \|S\|_{\mathrm{cb}},\end{aligned}$$

where the supremum is over all (f_{ij}) in $M_m(X')$, m arbitrary. Thus, $\|A\| = \|S\|_{\mathrm{cb}}$.

For the reverse inequality, consider $S: X' \to R_n$, $S(f) = (\Psi_1(f), ..., \Psi_n(f))$, $\Psi_i \in X''$, and $T: R_n \to Y$, $T((\lambda_1, ..., \lambda_n)) = \lambda_1 y_1 + \cdots + \lambda_n y_n$ with $\tilde{u} = TS$. Repeating the above arguments we see that $\|T\|_{\mathrm{cb}} = \|(y_1, ..., y_n)^{\mathrm{t}}\|$ in $C_n(Y)$, $\|S\|_{\mathrm{cb}} = \|(\Psi_1, ..., \Psi_n)\|$ in $R_n(X'')$, where X'' is the standard operator space second dual. Note that $u = \Psi_1 \otimes y_1 + \cdots + \Psi_n \otimes y_n$ in $X'' \otimes Y$. Thus $\|S\|_{\mathrm{cb}}\, \|T\|_{\mathrm{cb}} \geqslant \|u\|_{\mathrm{h}}$ in $X'' \otimes_{\mathrm{h}} Y$. But by the injectivity of the Haagerup norm and the fact that $X \subseteq X''$ completely isometrically (Theorem 2.11), we have that the right-hand side is larger than the left. ∎

Remark 3.12. From the proof of the above theorem we see that if X and Y are matrix normed but not operator spaces, then the factorization formula is smaller, in general, than $\|u\|_{\mathrm{h}}$. In fact, it is the Haagerup norm of u taken in $X'' \otimes_{\mathrm{h}} Y$. Recall that when X is not an operator space, then the canonical inclusion of X into the standard operator space dual X'' is completely contractive, but not a complete isometry.

Remark 3.13. It is possible to prove Theorem 3.11 from first principles, without using the injectivity of the Haagerup norm. It then follows from the Banach space argument for the injectivity of H that the Haagerup norm is injective. However, to follow this line of proof, one needs an additional fact, namely, that subspaces and matrix normed quotients of R_n are completely isometrically isomorphic to R_k, for some k. We prove this fact in Section 4. Recall that if X is matrix normed and $Y \subseteq X$, then X/Y is endowed with a matrix norm, via

$$\|(x_{ij}+Y)\| = \inf\{\|(x_{ij}+y_{ij})\| : y_{ij} \in Y\}.$$

4. Connections with the Commutative Theory

In his address to the ICM in 1986 E. G. Effros [Ef] proposed a theory of generalized or "quantized" functional analysis. The idea is that the category of normed vector spaces and bounded linear maps may be embedded as a subcategory in the category of operator spaces and completely bounded maps. More precisely if we are given a normed vector space X we may associate with it [ER1] a canonical commutative operator space $\mathrm{MIN}(X)$, as in Example 2.3. Thus normed vector spaces may be considered as the commutative operator spaces; and the matricial theory of general operator spaces is a "noncommutative" generalization of functional analysis. For instance, the extension theorem for completely bounded maps (see Section 2) is a generalization of the classical Hahn–Banach theorem. Ruan's characterization of operator spaces is a noncommutative Bourbaki–Alaoglu theorem [Ef, Ru]. The notion of the standard dual of an operator space (Sections 2 and 5. [Bl4]) leads to further generalizations. We now identify some other results in the completely bounded theory which are generalizations of known results in Banach space tensor product theory. This leads us into a discussion of Pisier's gamma norms [Pi2]. Given a normed space X and a family of sesquilinear forms on X we construct a natural embedding of X into the bounded operators on a Hilbert space. This construction is interesting in many ways; in particular it allows us to identify each gamma norm as a special case of the Haagerup norm.

In [Gr], Grothendieck considered 14 natural norms defined on the algebraic tensor product $X \otimes Y$ of two normed spaces X and Y. The most important ones for our purposes are the injective norm λ, the projective norm γ, and the norm H. The last-mentioned norm H, corresponding to factorization through a Hilbert space, is also known as γ_2 [Pi1] or α_{22} [GL]; and may be defined as

$$\|u\|_{\mathrm{H}} = \inf\left\{\sup\left\{\left(\sum_{i=1}^{n} |f(x_i)|^2\right)^{1/2} \left(\sum_{i=1}^{n} |g(y_i)|^2\right)^{1/2}\right\}\right\}$$

for $u \in X \otimes Y$, where the supremum is taken over all $f \in \mathrm{BALL}(X')$ and all $g \in \mathrm{BALL}(Y')$, and the infimum is taken over all representations $u = \sum_{i=1}^{n} x_i \otimes y_i$ of u.

We show in Section 5 that the spatial operator space tensor norm is the noncommutative generalization of the injective tensor norm λ, and we find the noncommutative generalization of the projective norm γ. We now proceed to find the norm which the Haagerup operator space tensor norm generalizes, from which follow some interesting observations.

If X and Y are two normed spaces we may regard them as commutative operator spaces in the way outlined above and form the Haagerup tensor product $\mathrm{MIN}(X) \otimes_{\mathrm{h}} \mathrm{MIN}(Y)$. This new operator space will not in general be a commutative operator space. Now consider $\mathrm{MIN}(X) \otimes_{\mathrm{h}} \mathrm{MIN}(Y)$ regarded as a normed vector space. This is the (uncompleted) tensor product of the normed spaces X and Y with respect to a certain normed space tensor norm, which we now identify.

PROPOSITION 4.1. *If X and Y are normed spaces then* $\mathrm{MIN}(X) \otimes_{\mathrm{h}} \mathrm{MIN}(Y)$ *is linearly isometrically isomorphic to* $X \otimes_{\mathrm{H}} Y$.

This result follows immediately by inspection. A proof is also contained in the discussion at the end of this section.

Once it is seen that the Haagerup norm $\|\cdot\|_{\mathrm{h}}$ is the noncommutative generalization of Grothendieck's H-norm many results in the theory of the Haagerup norm and completely bounded multilinear maps may be seen to be noncommutative generalizations of known results in the metric theory of tensor products of Banach spaces. For instance Paulsen and Smith's complete injectivity of the Haagerup norm (see Section 3) is the noncommutative generalization of the injectivity of the H-norm (a fact well known to Banach space experts). The characterization of operator algebras given in [BRS] is shown in [Bl3] to be the noncommutative version of Tonge's characterization of uniform algebras [Tn]. It follows from [Bl2, Proposition 2] that the canonical map from the completed Haagerup tensor product of two operator spaces into the completed spatial (or injective) tensor product of those spaces is a monomorphism; and this generalizes a result of Carne [Ca2]. Finally, the factorization results for the Haagerup norm in Section 3 generalize the factorization properties of H.

G. Pisier has pointed out to the authors that although the Haagerup norm is associative (Proposition 3.1) the H-norm is not (that is, $(X \otimes_{\mathrm{H}} Y) \otimes_{\mathrm{H}} Z \neq X \otimes_{\mathrm{H}} (Y \otimes_{\mathrm{H}} Z)$ isometrically for normed spaces X, Y, and Z). If one loses track of the categories involved then this may seem to be a paradox; the point is that $\mathrm{MIN}(X) \otimes_{\mathrm{H}} \mathrm{MIN}(Y)$ is not completely isometrically isomorphic to $\mathrm{MIN}(X \otimes_{\mathrm{H}} Y)$. This nonassociativity we believe is one piece of evidence that the H-norm should be regarded as a matricial norm (i.e., the restriction of the Haagerup norm) instead of purely as a normed space tensor norm.

In [Pi2] Pisier denotes the H-norm by γ_2 and introduces a whole class of related tensor norms which he calls *gamma norms*. We begin by recalling the definition of the gamma norms. Suppose X_1 and X_2 are normed spaces and suppose that K_1 (respectively K_2) are sets of positive sesquilinear forms on $X_1 \times X_1$ (respectively $X_2 \times X_2$) with norm less than or equal to 1. We shall need one further condition on the sets K_i ($i = 1, 2$), namely we require for $i = 1, 2$ that

$$\|x\| = \sup\{|\varphi(x, x)|^{1/2} : \varphi \in K_i\}$$

for each $x \in X_i$. This last condition is different from the condition Pisier assumes; but it is certainly not more restrictive.

We can now define a norm α on $X_1 \otimes X_2$ by

$$\alpha(u) = \inf\left\{\sup\left\{\left(\sum_{i=1}^{n} \varphi(x_i, x_i)\right)^{1/2} \left(\sum_{i=1}^{n} \psi(y_i, y_i)\right)^{1/2}\right\}\right\}$$

for $u \in X_1 \otimes X_2$, where the supremum is taken over $\varphi \in K_1$, and ψ in K_2, and the infimum is taken over all representations $u = \sum_{i=1}^{n} x_i \otimes y_i$ of u. Such a norm will be called a *gamma norm*. It is not clear yet that they are in fact norms; this will follow from the sequel.

EXAMPLE 4.2 [Pi2]. Let X_1 and X_2 be normed vector spaces, and for $i = 1, 2$ let K_i be the set of sesquilinear forms on $X_i \times X_i$ given by

$$(x, y) \to \varphi(x) \overline{\varphi(y)},$$

where φ is a contractive linear functional on X_i. The corresponding gamma norm is Grothendieck's norm H (or γ_2).

The example above shows that H, which by Proposition 4.1 is a special case of the Haagerup norm, is a gamma norm. We now show that all gamma norms are special cases of the Haagerup norm. To do this we need the following construction.

Let X be a normed space and let K be a set of positive sesquilinear forms on $X \times X$ of norm less than or equal to 1 satisfying the condition

$$\|x\| = \sup\{|\varphi(x, x)|^{1/2} : \varphi \in K\}$$

for each $x \in X$. We find it convenient to consider bilinear forms on $X \times X^*$ instead of sesquilinear forms on $X \times X$. Here X^* is the conjugate normed space of X; that is $X^* = \{x^* : x \in X\}$, where $\|x^*\| = \|x\|$, $x^* + y^* = (x + y)^*$, and $(\lambda x)^* = \bar{\lambda} x^*$. There is clearly a bijective correspondence between sesquilinear forms φ on $X \times X$, and bilinear maps $\tilde{\varphi}: X \times X^* \to \mathbb{C}$, namely

$$\tilde{\varphi}(x, y^*) = \varphi(x, y).$$

This correspondence $\varphi \to \tilde{\varphi}$ transforms the set K into a set $\tilde{K}$ of contractive bilinear maps on $X \times X^*$ which satisfy the condition

$$\tilde{\varphi}(x, x^*) \geqslant 0 \qquad \text{for each} \quad x \in X.$$

Moreover, for $x \in X$ we have

$$\|x\| = \sup\{|\tilde{\varphi}(x, x^*)| : \tilde{\varphi} \in \tilde{K}\}.$$

Now if $\tilde{\varphi} \in \tilde{K}$ then the n-fold amplification [PS] $\tilde{\varphi}_n: M_n(X) \times M_n(X^*) \to M_n$ is easily seen to satisfy an analogous condition

$$\tilde{\varphi}_n(x, x^*) \geqslant 0 \qquad \text{for} \quad x \in M_n(X),$$

where if $x = [x_{ij}]$ then $x^* = [x_{ji}^*]$.

We define a matricial structure on X by defining

$$\|x\|_n = \sup\{\|\tilde{\varphi}_n(x, x^*)\|^{1/2} : \tilde{\varphi} \in \tilde{K}\}$$

for $x \in M_n(X)$. We show $\|\cdot\|_n$ is a norm by considering the sesquilinear forms on $M_n(X) \times M_n(X)$ given by

$$(x, y) \to \langle \tilde{\varphi}_n(x, y^*)\underline{\zeta}, \underline{\zeta} \rangle$$

for $\tilde{\varphi} \in \tilde{K}$ and $\underline{\zeta} \in \mathrm{BALL}(C_n)$. Each of these induces a seminorm on $M_n(X)$, and since the supremum of these seminorms is $\|\cdot\|_n$ it follows that $\|\cdot\|_n$ is a seminorm. However, if $\|x\|_n = 0$ then for $\tilde{\varphi} \in \tilde{K}$ and $\underline{\zeta} \in \mathrm{BALL}(C_n)$ we have

$$\begin{aligned} 0 &= \langle \tilde{\varphi}_n(x, x^*)\underline{\zeta}, \underline{\zeta} \rangle \\ &= \sum_{i,j,k=1}^{n} \bar{\zeta}_i \tilde{\varphi}(x_{ik}, x_{jk}^*)\, \zeta_j \\ &= \sum_{k=1}^{n} \tilde{\varphi}\left(\sum_{i=1}^{n} \bar{\zeta}_i x_{ik}, \left(\sum_{i=1}^{n} \bar{\zeta}_i x_{ik} \right)^* \right). \end{aligned}$$

Thus for each k

$$\left\| \sum_{i=1}^{n} \bar{\zeta}_i x_{ik} \right\| = \sup\left\{ \tilde{\varphi}\left(\sum_{i=1}^{n} \bar{\zeta}_i x_{ik}, \left(\sum_{i=1}^{n} \bar{\zeta}_i x_{ik} \right)^* \right) : \tilde{\varphi} \in \tilde{K} \right\} = 0.$$

Choosing $\underline{\zeta}$ appropriately yields $x = 0$; and so $\|\cdot\|_n$ is a norm.

We now show that these matrix norms satisfy Ruan's conditions [Ru], so that $(X, \|\cdot\|_n)$ is an operator space.

It is clear that $\|x \oplus y\|_{n+m} = \max\{\|x\|_n, \|y\|_m\}$ for $x \in M_n(X)$ and $y \in M_m(X)$. Now if $A, B \in M_n$ and if $x \in M_n(X)$ we have

$$\begin{aligned}\|Ax\|_n &= \sup\{\|\tilde{\varphi}_n(Ax, (Ax)^*)\|^{1/2} : \tilde{\varphi} \in \tilde{K}\} \\ &= \sup\{\|\tilde{\varphi}_n(Ax, x^*A^*))\|^{1/2} : \tilde{\varphi} \in \tilde{K}\} \\ &= \sup\{\|A\tilde{\varphi}_n(x, x^*)\, A^*\|^{1/2} : \tilde{\varphi} \in \tilde{K}\} \\ &\leqslant \|A\|\ \|x\|_n,\end{aligned}$$

and

$$\begin{aligned}\|xB\|_n &= \sup\{\|\tilde{\varphi}_n(xB, (xB)^*)\|^{1/2} : \tilde{\varphi} \in \tilde{K}\} \\ &= \sup\{\|\tilde{\varphi}_n(xB, B^*x^*)\|^{1/2} : \tilde{\varphi} \in \tilde{K}\} \\ &\leqslant \|B\| \sup\{\|\tilde{\varphi}_n(x, x^*)\|^{1/2} : \tilde{\varphi} \in \tilde{K}\} \\ &= \|B\|\ \|x\|_n.\end{aligned}$$

To prove the last inequality we have employed the following fact:

$$\begin{aligned}\|B\|^2\, \tilde{\varphi}_n(x, x^*) - \tilde{\varphi}_n(xB, B^*x^*) &= \|B\|^2\, \tilde{\varphi}_n(x, x^*) - \tilde{\varphi}_n(x, BB^*x^*) \\ &= \tilde{\varphi}_n(x, (\|B\|^2 - BB^*)\, x^*) \\ &= \tilde{\varphi}_n(x(\|B\|^2 - BB^*)^{1/2}, (\|B\|^2 - BB^*)^{1/2}\, x) \\ &\geqslant 0.\end{aligned}$$

Therefore $(X, \|\cdot\|_n)$ is an operator space. For notational convenience we shall write $\mathscr{R}_K(X)$ for this space. We denote the opposite operator space $\mathscr{R}_K(X)^{\mathrm{op}}$ by $\mathscr{L}_K(X)$. In fact, $\mathscr{L}_K(X)$ can be constructed directly by considering bilinear forms on $X^* \times X$ instead of on $X \times X^*$.

EXAMPLE 4.3. Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space, and let $K = \{\langle\cdot,\cdot\rangle\}$. Then $\mathscr{R}_K(X)$ and $\mathscr{L}_K(X)$ are "generalized" inner product spaces. These operator spaces behave well with respect to taking subspaces and quotients. For instance, if $X = l_n^2$ it is not hard to see that $\mathscr{R}_K(X) = R_n$ and $\mathscr{L}_K(X) = C_n$ completely isometrically. This shows that the operator space structure of R_n or C_n is "basis free." It also follows that subspaces and operator space quotients of R_n (respectively C_n) are completely isometrically isomorphic to R_k (respectively C_k) for some $k \leqslant n$. Note that if we form the tensor product $l_n^2 \otimes_\alpha l_n^2$ with respect to the gamma norm α given by the natural sesquilinear form then we obtain M'_n.

EXAMPLE 4.4. Let X be a normed vector space and let K be the set of sesquilinear forms on X of the type described in Example 4.2; then $\mathscr{L}_K(X) = \mathscr{R}_K(X) = \mathrm{MIN}(X)$ completely isometrically.

EXAMPLE 4.5. Let X be an operator space contained in a C^*-algebra $\mathscr{A}$, and let $\mathscr{S}$ be the set of sesquilinear forms on $X \times X$ of the form

$$(x, y) \to s(xy^*)$$

for states s of $\mathscr{A}$. Let $\mathscr{F}$ be the set of sesquilinear forms on $X \times X$ of the form

$$(x, y) \to s(y^*x)$$

for states s of $\mathscr{A}$. Then we can form the operator spaces $\mathscr{L}_{\mathscr{S}}(X)$, $\mathscr{R}_{\mathscr{S}}(X)$, $\mathscr{L}_{\mathscr{F}}(X)$, and $\mathscr{R}_{\mathscr{F}}(X)$ in the manner described above; in general none of these four spaces are completely isometrically isomorphic to X. In fact, in general, X is not completely isometrically isomorphic to $\mathscr{L}_K(X)$ or $\mathscr{R}_K(X)$ for any set K of sesquilinear forms.

We now return to the gamma norms. If X_1 and X_2 and K_1 and K_2 are as in the definition of the gamma norm then we may form the operator spaces $\mathscr{R}_{K_1}(X_1)$ and $\mathscr{L}_{K_2}(X_2)$. An easy calculation shows that $\mathscr{R}_{K_1}(X_1) \otimes_{\mathrm{h}} \mathscr{L}_{K_2}(X_2)$ is isometrically isomorphic to $X_1 \otimes_\alpha X_2$, where α is the gamma norm defined earlier in terms of K_1 and K_2. Thus each gamma norm is indeed a special case of the Haagerup norm.

There is a partial converse to this. Pisier has pointed out [Pi2] that the Haagerup tensor product $X_1 \otimes_{\mathrm{h}} X_2$ of two operator spaces X_1 and X_2 is isometrically isomorphic to the tensor product $X_1 \otimes_\alpha X_2$ with respect to the gamma norm α defined by the set $\mathscr{S}$ of sesquilinear forms defined on $X \times X$ and the set $\mathscr{F}$ of sesquilinear forms defined on $Y \times Y$ in Example 4.5. In the light of the preceding discussion we see $\mathscr{R}_{\mathscr{S}}(X_1) \otimes_{\mathrm{h}} \mathscr{L}_{\mathscr{F}}(X_2)$ is isometrically isomorphic to $X_1 \otimes_{\mathrm{h}} X_2$ since both are isometrically isomorphic to $X_1 \otimes_\alpha X_2$. However, since in general $\mathscr{R}_{\mathscr{S}}(X_1)$ and $\mathscr{L}_{\mathscr{F}}(X_2)$ are not completely isometrically isomorphic to X_1 and X_2, respectively, $R_{\mathscr{S}}(X_1) \otimes_{\mathrm{h}} \mathscr{L}_{\mathscr{F}}(X_2)$ and $X_1 \otimes_{\mathrm{h}} X_2$ are not completely isometrically isomorphic.

Finally, we note that if we consider *operator valued* sesquilinear maps instead of scalar valued forms a similar construction is possible, and moreover all operator spaces arise in this manner.

5. OPERATOR SPACE TENSOR NORMS

In this section we axiomatize the concept of an operator space tensor norm. The theory of C^*-algebra tensor norms is now rich and well developed, playing a significant role in the structure theory of C^*-algebras [La]. Similarly operator algebra tensor norms [PP] and operator system tensor norms [CE] have been studied, with applications to representation theory. Operator space tensor norms on the other hand have been neglected, with interest concentrated on two particular operator space tensor norms, the spatial and the Haagerup norms. One reason for this has been the absence of an appropriate duality theory for operator spaces. However, armed with the concept of the standard dual to an operator space a natural definition of an operator space tensor norm, and corresponding theory, presents itself. Moreover this theory has many analogies with the tensor product theory for Banach spaces.

We therefore begin with a review of the elementary facts from the Banach space theory. It is convenient however to state the results for general normed spaces.

If X and Y are normed spaces and if α is a norm defined on the algebraic tensor product $X \otimes Y$ then α is said to be a cross norm provided $\alpha(x \otimes y) = \|x\| \, \|y\|$ for $x \in X$ and $y \in Y$. We will write $X \otimes_\alpha Y$ for the normed space $(X \otimes Y, \alpha)$. It is clear that $X \otimes_\alpha \mathbb{C}$ and $\mathbb{C} \otimes_\alpha X$ are isometrically isomorphic to X for any cross norm α.

There is a natural duality pairing between $X' \otimes Y'$ and $X \otimes Y$ given by

$$\langle f \otimes g, x \otimes y \rangle = f(x)\, g(y)$$

for $f \in X'$, $g \in Y'$, $x \in X$, and $y \in Y$. Now if α is a norm on $X \otimes Y$ the pairing above provides $X' \otimes Y'$ with a natural norm, which we write as α'.

If α and β are norms on $X \otimes Y$ we write $\alpha \leqslant \beta$ if $\alpha(u) \leqslant \beta(u)$ for each $u \in X \otimes Y$. With respect to this ordering there is a greatest cross norm γ, the *projective* tensor norm. However, in general there is no least cross norm. Nonetheless, there is a least cross norm α such that the dual norm α' is also a cross norm, namely the *injective* tensor norm λ. For an element $\sum_{i=1}^n x_i \otimes y_i$ of $X \otimes Y$ we have

$$\begin{aligned}\lambda\left(\sum_{i=1}^n x_i \otimes y_i\right) \\ &= \sup\left\{\left|\sum_{i=1}^n f(x_i)\, g(y_i)\right| : f \in \mathrm{BALL}(X'),\ g \in \mathrm{BALL}(Y')\right\}.\end{aligned}$$

We can identify $X \otimes_\lambda Y$ with a subspace of $B(X', Y)$ via the isometric embedding $X \otimes_\lambda Y \to B(X', Y)$ which takes an element $\sum_{i=1}^n x_i \otimes y_i$ to the operator which maps

$$f \to \sum_{i=1}^n f(x_i)\, y_i$$

for $f \in X'$. Similarly $X \otimes_\lambda Y$ may be identified as a subspace of $B(Y', X)$.

To describe $X \otimes_\gamma Y$ we identify its dual space. We recall that if X, Y, and Z are normed spaces and if $\Psi: X \times Y \to Z$ is a bilinear map then Ψ is said to be *bounded* if $\|\Psi\| = \sup\{\|\Psi(x, y)\| : x \in \mathrm{BALL}(X),\ y \in \mathrm{BALL}(Y)\} < \infty$. We write $B(X \times Y; Z)$ for the resulting normed space of all bounded bilinear maps $X \times Y \to Z$. It is not hard to see that $B(X \times Y; Z)$ and $B(X, B(Y, Z))$ and $B(Y, B(X, Z))$ are isometrically isomorphic via canonical identifications. Now $X \otimes_\gamma Y$ is defined so that $(X \otimes_\gamma Y)'$ is isometrically isomorphic to $B(X \times Y; \mathbb{C})$ via the usual identifications of bilinear maps with linear maps on the tensor product.

From the above one can show that if α is a norm on $X \otimes Y$ then $\lambda \leqslant \alpha \leqslant \gamma$ if and only if α and α' are cross norms. Also, on $X' \otimes Y'$ we have $\gamma' = \lambda$ although in general $\lambda' \neq \gamma$.

We now establish the analogous results for tensor products of operator spaces. Suppose X and Y are operator spaces, and that $x = [x_{ij}] \in M_n(X)$ and $y = [y_{k\ell}] \in M_m(Y)$. It is convenient when there is no danger of confusion to write $x \otimes y$ for the element $[x_{ij} \otimes y_{kl}]_{(i,k),(j,\ell)}$ of $M_{nm}(X \otimes Y)$; in other words we employ the algebraic identification of $M_n(X) \otimes M_m(Y)$ with $M_{nm}(X \otimes Y)$. Suppose now that for each $n \in \mathbb{N}$ there is a norm α_n defined on $M_n(X \otimes Y)$ such that $(X \otimes Y, \alpha_n)$ is an operator space. We say that α is an *operator space cross norm* on $X \otimes Y$ if

$$\alpha_{nm}(x \otimes y) = \|x\|_n \, \|y\|_m$$

for each $x \in M_n(X)$ and $y \in M_m(Y)$.

We will write $X \otimes_\alpha Y$ for the operator space $(X \otimes Y, \alpha_n)$. It is clear that if X is an operator space then $X \otimes_\alpha \mathbb{C}$ (respectively $\mathbb{C} \otimes_\alpha X$) is completely isometrically isomorphic to X for any operator space cross norm α on $X \otimes \mathbb{C}$ (respectively $\mathbb{C} \otimes X$).

If $(X \otimes Y, \alpha_n)$ is a matrix normed space then the natural duality pairing between $X' \otimes Y'$ and $X \otimes Y$ provides $X' \otimes Y'$ with an operator space structure $(X' \otimes Y', \alpha'_n)$. Explicitly

$$\alpha'_n([\Psi_{ij}]) = \sup\{\|[\langle \Psi_{ij}, U_{k\ell} \rangle]\| : [U_{k\ell}] \in \mathrm{BALL}(M_m(X \otimes_\alpha Y))\}$$

for $[\Psi_{ij}] \in M_n(X' \otimes Y')$. In the future if E and F are matrix normed spaces and $\langle \cdot, \cdot \rangle: F \times E \to \mathbb{C}$ is a bilinear map we shall write an expression $[\langle f_{ij}, e_{kl} \rangle]$ as $\langle f, e \rangle_{nm}$, or simply $\langle f, e \rangle$ (where $f = [f_{ij}] \in M_n(F)$ and $e = [e_{k\ell}] \in M_m(E)$). Thus the dual norm α' may be written as

$$\alpha'_n(\Psi) = \sup\{\|\langle \Psi, U \rangle\| : U \in \mathrm{BALL}(M_m(X \otimes_\alpha Y))\}$$

for $\Psi \in M_n(X' \otimes Y')$.

If $(X \otimes Y, \alpha_n)$ and $(X \otimes Y, \beta_n)$ are matrix normed spaces we write $\alpha \leqslant \beta$ if $\alpha_n(u) \leqslant \beta_n(u)$ for all n and for all $u \in M_n(X \otimes Y)$. We shall show presently that with respect to this ordering there is a greatest operator space cross norm, the *projective operator space cross norm* $\wedge$. In general there is no least operator space cross norm; nonetheless we show that there is a least operator space cross norm α such that the dual norm α' is also an operator space cross norm. This is the *spatial* or *injective* [Pa1] operator space cross norm $\vee$, defined via the inclusion of $X \otimes Y$ in $\mathscr{B}(\mathscr{H} \otimes \mathscr{K})$ if X and Y as operator spaces are contained in $\mathscr{B}(\mathscr{H})$ and $\mathscr{B}(\mathscr{K})$, respectively. We may write the norm $\vee$ more explicitly as

$$\vee_n (U) = \sup\{\|(S \otimes T)^{(n)} (U)\|\}$$

for $U \in M_n(X \otimes Y)$, where the supremum is taken over all Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$ and all completely contractive maps $S: X \to B(\mathscr{H})$ and $T: Y \to B(\mathscr{K})$, and where the norm in the right-hand side is calculated in $M_n(B(\mathscr{H} \otimes \mathscr{K}))$.

The next proposition shows that it suffices to consider finite dimensional Hilbert spaces in the (second) definition of $\vee$ above.

THEOREM 5.1. *Let X and Y be operator spaces. The injective operator space norm is given by*

$$\vee_n (U) = \sup\{\|\langle f \otimes g, U\rangle_{nm}\|\}$$

for $U \in M_n(X \otimes Y)$, where the supremum is taken over all $f \in \mathrm{BALL}(M_p(X'))$ and $g \in \mathrm{BALL}(M_q(Y'))$.

Proof. Using the canonical identifications of $M_p(X')$ and $M_q(Y')$ with $\mathrm{CB}(X, M_p)$ and $\mathrm{CB}(Y, M_q)$, respectively, it is seen that the right-hand side of the equality above equals the quantity

$$\sup\{\|(S \otimes T)^{(n)} (U)\|\}$$

for $U \in M_n(X \otimes Y)$, where the supremum is taken over all *finite dimensional* Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, and all S, T as before. It is clear that this quantity is dominated by $\vee_n (U)$. We now use the same technique as that in the proof of the complete inclusion of an operator space in its second dual (Theorem 2.11) to obtain the reverse inequality. Suppose that $X \subseteq \mathscr{B}(\mathscr{H})$ and $Y \subseteq \mathscr{B}(\mathscr{K})$, where $\mathscr{H}$ and $\mathscr{K}$ now are not necessarily finite dimensional Hilbert spaces. Then $\vee_n (U) = \|U\|_{M_n(\mathscr{B}(\mathscr{H} \otimes \mathscr{K}))}$. Let $\mathscr{F}$ be the family of all finite dimensional subspaces F of $\mathscr{H}$, and let $\mathscr{G}$ be the family of all finite dimensional subspaces G of $\mathscr{K}$. For $F \in \mathscr{F}$ (respectively $G \in \mathscr{G}$) write φ_F (respectively ψ_G) for the compression map $\mathscr{B}(\mathscr{H}) \to \mathscr{B}(F)$ (respectively $B(\mathscr{K}) \to B(G)$). Then it is easy to see that

$$\|U\|_{M_n(B(\mathscr{H} \otimes \mathscr{K}))} = \sup\{\|(\varphi_F \otimes \psi_G)^{(n)} (U)\|_{M_n(B(F \otimes G))}\},$$

where the supremum is taken over all $F \in \mathscr{F}$ and $G \in \mathscr{G}$. Since each φ_F and ψ_G is completely contractive the result follows. ∎

Just as in the normed space case there are the usual canonical maps from $X \otimes Y$ into the space of linear operators from X' to Y (respectively Y' to X). It is not hard to see that the ranges of these maps are composed of completely bounded operators.

COROLLARY 5.2. *Let X and Y be operator spaces. The canonical maps from $X \otimes_\vee Y$ into* $\mathrm{CB}(X', Y)$ *and* $\mathrm{CB}(Y', X)$, *respectively, are complete isometries.*

Proof. This follows directly from Theorems 5.1 and 2.11. ∎

To describe the projective operator space cross norm $\wedge$ we identify the corresponding dual space.

DEFINITION 5.3. Let X, Y, and Z be matrix normed spaces, and let $\Psi: X \times Y \to Z$ be a bilinear map. We say that Ψ is *jointly completely bounded* if

$$\|\Psi\| = \sup\{\|[\Psi(x_{ij}, y_{k\ell})]\| : [x_{ij}] \in \mathrm{BALL}(M_p(X)), [y_{k\ell}] \in \mathrm{BALL}\ M_q(y))\}$$

is finite.

The space $\mathrm{JCB}(X \times Y, Z)$ of jointly completely bounded maps $X \times Y \to Z$ has a natural matrix normed structure, given by identifying $M_n(\mathrm{JCB}(X \times Y, Z))$ with $\mathrm{JCB}(X \times Y, M_n(Z))$. If $\Psi = [\Psi_{rs}] \in M_n(\mathrm{JCB}(X \times Y, Z))$ then

$$\|\Psi\|_n = \sup\{\|[\Psi_{rs}(x_{ij}, y_{k\ell})]\| :$$
$$[x_{ij}] \in \mathrm{BALL}(M_p(X)), [y_{kl}] \in \mathrm{BALL}(M_q(Y))\}.$$

With this matrix norm structure it is easy to see that $\mathrm{JCB}(X \times Y, Z)$ and $\mathrm{CB}(X, \mathrm{CB}(Y, Z))$ and $\mathrm{CB}(Y, \mathrm{CB}(X, Z))$ are completely isometrically isomorphic via canonical identifications. Now $X \otimes_\wedge Y$ is defined so that $(X \otimes_\wedge Y)'$ is completely isometrically isomorphic to $\mathrm{JCB}(X \times Y, \mathbb{C})$ via the usual identification of bilinear maps with linear maps on the tensor product. Henceforth we shall often ignore the distinction between bilinear maps and linear maps on the tensor product. More specifically, for $U \in M_n(X \otimes Y)$ define

$$\wedge_n(U) = \sup\{\|\langle \Psi, U \rangle\|\},$$

where the supremum is taken over all $\Psi \in \mathrm{BALL}(M_m(\mathrm{JCB}(X \times Y, \mathbb{C})))$. It is not difficult to see using Ruan's theorem [Ru] (or directly as in [Bl4]) that with this matrix norm structure $X \otimes_\wedge Y$ is an operator space.

PROPOSITION 5.4. *If X and Y are operator spaces then $(X \otimes_\wedge Y)'$ is canonically completely isometrically isomorphic to each of the spaces* $\mathrm{JCB}(X \times Y, \mathbb{C})$, $\mathrm{CB}(X, Y')$ *and* $\mathrm{CB}(Y, X')$.

Proof. It is clear that if $\Psi \in \mathrm{BALL}(M_m(\mathrm{JCB}(X \times Y, \mathbb{C})))$ then $\Psi \in \mathrm{BALL}(M_m((X \otimes_\wedge Y)'))$. Conversely, if $\Psi \in \mathrm{BALL}(M_m((X \otimes_\wedge Y)'))$ then, since $x \otimes y \in \mathrm{BALL}(M_{pq}(X \otimes_\wedge Y))$ for $x \in \mathrm{BALL}(M_p(X))$ and $y \in \mathrm{BALL}(M_q(Y))$, we have

$$\|\langle \Psi, x \otimes y \rangle\| \leqslant \|x\|_p \, \|y\|_q;$$

therefore $\Psi \in \mathrm{BALL}(M_m(\mathrm{JCB}(X \times Y, \mathbb{C})))$. ∎

THEOREM 5.5. *The projective operator space tensor norm is an operator space cross norm. If X and Y are operator spaces and if α is an operator space cross norm on $X \otimes Y$ then $\alpha_n \leqslant \wedge_n$ on $M_n(X \otimes Y)$ for each n. Thus the projective operator space tensor norm is the largest operator space cross norm.*

Proof. We begin by proving the second statement; this together with the fact that the spatial tensor norm is an operator space cross norm gives the first statement.

If α is an operator space cross norm on $X \otimes Y$ and if $\Psi \in \mathrm{BALL}(M_m((X \otimes_\alpha Y)')$ then for $x \in \mathrm{BALL}(M_p(X))$ and $y \in \mathrm{BALL}(M_q(Y))$ we have

$$\|\langle \Psi, x \otimes y \rangle\| \leqslant 1.$$

Therefore Ψ is jointly completely contractive, and so by Theorem 2.11 we have

$$\begin{aligned} \alpha_n(U) &= \sup\{\|\langle \Psi, U \rangle\| : \Psi \in \mathrm{BALL}(M_m((X \otimes_\alpha Y)')\} \\ &\leqslant \sup\{\|\langle \Psi, U \rangle\| : \Psi \in \mathrm{BALL}(M_m((X \otimes_\wedge Y)')\} \\ &= \wedge_n (U), \end{aligned}$$

for $U \in M_n(X \otimes Y)$. ∎

THEOREM 5.6. *The dual $\wedge'$ of the projective operator space cross norm is the spatial norm $\vee$.*

Proof. Let X and Y be operator spaces, and let $\Psi \in M_m(X' \otimes Y')$. We wish to show that the norm of Ψ as an element of $M_m((X \otimes_\wedge Y)')$ coincides with its norm as an element of $M_m(X' \otimes_\vee Y')$. Consider the diagram

$$\begin{tikzcd}[column sep=small]
 & M_m(X' \otimes_\vee Y') \arrow[rr] \arrow[dl] & & M_m((X \otimes_\wedge Y)') \arrow[dr] & \\
M_m(\mathrm{CB}(X'', Y')) \arrow[drr] & & & & M_m(\mathrm{CB}(X, Y')) \arrow[dll] \\
 & & M_m(\mathrm{CB}(X'', Y''')) & &
\end{tikzcd}$$

where all the maps are the canonical ones. The only map possibly needing explanation is the map on the lower right: this is the amplification of the second Banach space adjoint map $''$: $\mathrm{CB}(X, Y') \to \mathrm{CB}(X'', Y''')$. This map is easily seen to be a complete isometry; therefore all the maps in the diagram are isometries except possibly for the map on the top. If we could show that the diagram is commutative for $n = 1$ then it is commutative for all n, and this would complete the proof.

However, if $\Psi \in X' \otimes Y'$, $\Psi = \sum_{i=1} f_i \otimes g_i$ say, then composing clockwise we obtain first the map $\sum_{i=1}^{n} f_i(\cdot)\, g_i$ in $\mathrm{CB}(X, Y')$, and then the map $F \to \sum_k F(f_k)\, \hat{g}_k$ in $\mathrm{CB}(X'', Y''')$. Composing anticlockwise we first obtain the map $F \to \sum_k F(f_k)\, g_k$ in $\mathrm{CB}(X'', Y')$ and then the required map in $\mathrm{CB}(X'', Y''')$. ∎

THEOREM 5.7. *Let X and Y be operator spaces and suppose $(X \otimes Y, \alpha_n)$ is an operator space. Then $\vee \leqslant \alpha \leqslant \wedge$ if and only if both α and α' are operator space cross norms.*

Proof. Suppose α and α' are operator space cross norms; then $\alpha \leqslant \wedge$ by Theorem 5.5. Since α' is an operator space cross norm Theorem 5.1 implies that $\alpha \geqslant \vee$.

Now suppose that $\vee \leqslant \alpha \leqslant \wedge$; automatically α is an operator space cross norm. Since $\alpha \geqslant \vee$ it follows that $\alpha' \leqslant \vee'$, and Theorem 5.1 implies that $\vee'_{pq}(f \otimes g) \leqslant \|f\|_p \, \|g\|_q$ for $f \in M_p(X')$ and $g \in M_q(Y')$. On the other hand since $\alpha \leqslant \wedge$ we see that $\wedge' \leqslant \alpha'$, and Theorem 5.6 completes the proof. ∎

Remark 1. Since $\vee \leqslant \|\cdot\|_h \leqslant \wedge$ the Haagerup norm is an operator space cross norm.

Remark 2. Because of the symmetric nature of the constructions we see that $X \otimes_\alpha Y$ is canonically completely isometrically isomorphic to $Y \otimes_\alpha X$ for α the projective or spatial operator space tensor norms. Also, it is not hard to show that $(X \otimes_\alpha Y) \otimes_\alpha Z$ is canonically isometrically isomorphic to $X \otimes_\alpha (Y \otimes_\alpha Z)$ for α again as above. In fact the result above for $\alpha = \wedge$

follows from the fact (again not difficult to prove) that $\mathrm{CB}(X \otimes_{\wedge} Y, Z)$ is canonically isometrically isomorphic to $\mathrm{JCB}(X \times Y, Z)$.

Finally, we note that $\wedge_1$ is *not* in general equal to the projective normed space tensor norm γ even when restricted to commutative operator spaces. However, if X *or* Y is an operator space of the form $\mathrm{MAX}(E)$ for some normed space E then on $X \otimes Y$ these two norms do coincide. Indeed $\mathrm{MAX}(E) \otimes_{\wedge} \mathrm{MAX}(F) = \mathrm{MAX}(E \otimes_{\gamma} F)$.

It is sometimes convenient to consider a tensor norm as a functor in the following way [Ca1, Ca2, Mi, Gr, Sc]:

DEFINITION 5.8. A *uniform normed space tensor norm* α is an assignment of a normed space $X \otimes_{\alpha} Y$ to each pair (X, Y) of normed spaces, and an assignment of a bounded linear map

$$S \otimes_{\alpha} T\colon X_1 \otimes_{\alpha} Y_1 \to X_2 \otimes_{\alpha} Y_2$$

to each pair of bounded linear maps $S\colon X_1 \to X_2$ and $T\colon Y_1 \to Y_2$ such that:

(i) $X \otimes_{\alpha} Y$ is the algebraic tensor product $X \otimes Y$, together with a norm on $X \otimes Y$ which we write as α or $\|\cdot\|_{\alpha}$;

(ii) $S \otimes_{\alpha} T\colon X_1 \otimes_{\alpha} Y_1 \to X_2 \otimes_{\alpha} Y_2$ is the map

$$S \otimes_{\alpha} T(x_1 \otimes y_1) = S(x_1) \otimes T(y_1),$$

and the map

$$\begin{aligned} \otimes_{\alpha}\colon B(X_1, X_2) \times B(Y_1, Y_2) &\to B(X_1 \otimes_{\alpha} Y_1, X_2 \otimes_{\alpha} Y_2) \\ (S, T) &\to S \otimes_{\alpha} T \end{aligned}$$

is contractive; that is,

$$\|S \otimes_{\alpha} T\| \leqslant \|S\| \, \|T\|;$$

(iii) $\mathbb{C} \otimes_{\alpha} \mathbb{C} = \mathbb{C}$.

Together these conditions imply that $\lambda \leqslant \alpha \leqslant \gamma$ on $X \otimes Y$. The natural norms of Grothendieck [Gr] are all uniform tensor norms in the above sense.

In [Bl2] the first author gave an analoguous definition for C^*-algebra tensor norms and studied the implications of such a definition. We now do the same for operator space tensor norms.

DEFINITION 5.9. A *uniform operator space tensor norm* α is the assignment of an operator space $X \otimes_\alpha Y$ to each pair (X, Y) of operator spaces, and an assignment of a completely bounded linear map

$$S \otimes_\alpha T: X_1 \otimes_\alpha Y_1 \to X_2 \otimes_\alpha Y_2$$

to each pair of completely bounded linear maps $S: X_1 \to X_2$ and $T: Y_1 \to Y_2$ such that

(i) $X \otimes_\alpha Y$ is the algebraic tensor product $X \otimes Y$, together with a matrix norm structure on $X \otimes Y$ which we write as α_n or $\|\cdot\|_{\alpha_n}$;

(ii) $S \otimes_\alpha T: X_1 \otimes_\alpha Y_1 \to X_2 \otimes_\alpha Y_2$ is the map

$$S \otimes_\alpha T(x_1 \otimes y_1) = S(x_1) \otimes T(y_1),$$

and the map

$$\otimes_\alpha: \mathrm{CB}(X_1, X_2) \times \mathrm{CB}(Y_1, Y_2) \to \mathrm{CB}(X_1 \otimes_\alpha Y_1, X_2 \otimes_\alpha Y_2)$$
$$(S, T) \to S \otimes_\alpha T$$

is jointly completely contractive; that is,

$$\|[S_{ij} \otimes_\alpha T_{k\ell}]\|_{\mathrm{cb}} \leqslant \|[S_{ij}]\|_{\mathrm{cb}} \, \|[T_{k\ell}]\|_{\mathrm{cb}}$$

for $S_{ij} \in \mathrm{CB}(X_1, X_2)$ and $T_{k\ell} \in \mathrm{CB}(Y_1, Y_2)$;

(iii) $\mathbb{C} \otimes_\alpha \mathbb{C} = \mathbb{C}$.

If X and Y are not closed then we require also that $X \otimes_\alpha Y \subseteq \bar{X} \otimes_\alpha \bar{Y}$ completely isometrically.

PROPOSITION 5.10. *If α is a uniform operator space tensor norm then $\vee \leqslant \alpha \leqslant \wedge$ on the tensor product $X \otimes Y$ of two operator spaces X and Y.*

Proof. Putting $X_1 = Y_1 = \mathbb{C}$, $X_2 = X$, and $Y_2 = Y$ in (ii) we obtain

$$\alpha_{mn}(x \otimes y) \leqslant \|x\|_m \, \|y\|_n$$

for $x \in M_m(X)$ and $y \in M_n(Y)$; consequently $\alpha \leqslant \wedge$, as in Theorem 5.5.

On the other hand putting $X_1 = X$, $Y_1 = Y$, and $X_2 = Y_2 = \mathbb{C}$ in (ii) gives

$$\alpha'_{mn}(\varphi \otimes \psi) \leqslant \|\varphi\|_m \, \|\psi\|_n$$

for $\varphi \in M_m(X')$ and $\psi \in M_n(Y')$, and Theorem 5.1 now shows that $\alpha \geqslant \vee$.

PROPOSITION 5.11. *The spatial, projective, and Haagerup operator space tensor norms are uniform operator space tensor norms.*

Unfortunately we have no space here to prove the assertions of Proposition 5.11. The proofs are elementary but tedious to write down.

There are many other uniform operator space tensor norms which one can construct, and no doubt an investigation of the natural norms following Grothendieck's program [Gr, GL] would be interesting and rewarding.

We end this section by describing some further consequences of uniformity. The statements follow immediately from results in [Bl2].

THEOREM 5.12. *Let α be a uniform operator space tensor norm, and suppose α is an algebra norm on $\mathscr{A} \otimes \mathscr{B}$ for all C^*-algebras $\mathscr{A}$ and $\mathscr{B}$. Then the following are equivalent*

(i) $\overline{\mathscr{A} \otimes_\alpha \mathscr{B}}$ *is a C^*-algebra for all C^*-algebras $\mathscr{A}$ and $\mathscr{B}$.*

(ii) $\overline{\mathscr{A}_0 \otimes_\alpha \mathscr{B}_0}$ *is a C^*-algebra for some pair of nontrivial C^*-algebras $\mathscr{A}_0$ and $\mathscr{B}_0$.*

(iii) $\ell_2^\infty \otimes_\alpha \ell_2^\infty$ *is a C^*-algebra.*

COROLLARY 5.13. *The completed Haagerup tensor product $\overline{\mathscr{A} \otimes_\alpha \mathscr{B}}$ of two C^*-algebras $\mathscr{A}$ and $\mathscr{B}$ is a C^*-algebra if and only if $\mathscr{A}$ or $\mathscr{B}$ equals $\mathbb{C}$.*

THEOREM 5.14. *If $\mathscr{A}$ is a nuclear C^*-algebra then the canonical map*

$$\overline{\mathscr{A} \otimes_\alpha E} \to \overline{\mathscr{A} \otimes_\vee E}$$

is one to one for all operator spaces E and all uniform operator space tensor norms α.

Remark. Theorems 5.12 and 5.14 are true under much weaker hypotheses on the norm α (see [Bl2]).

Conjecture. The spatial tensor norm is the only uniform operator space tensor norm which preserves C^*-algebras, that is, which satisfies condition 5.12(i).

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